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Fluctuations in a Fluid under a Stationary Heat Flux. I. General Theory

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We present the basic formulas for a unified treatment of the correlation functions of the hydrodynamic variables in a fluid between two horizontal plates which is exposed to a stationary heat flux in the presence of a gravity field (Rayleigh–Bénard system). Our analysis is based on fluctuating hydrodynamics. In this paper (I) we show that in the nonequilibrium stationary state the hydrodynamic fluctuations evolve on slow and fast time scales that are widely separated. A time scale perturbation theory is used to diagonalize the hydrodynamic operator partially. This enables us to derive the eigenvalue equations for the nonequilibrium hydrodynamic modes. Therein we take into account the variation of the macroscopic quantities with position. The correlation functions are formally expressed in terms of the nonequilibrium modes. In paper II the slow hydrodynamic modes (viscous and viscoheat modes) will be determined explicitly for ideal heat-conducting plates with stick boundary conditions and used to compute the slow part of the correlation functions; in paper III the fast hydrodynamic modes (sound modes) will be explicitly determined for stick boundary conditions and used to compute the fast part of the correlation functions. In these papers we will also compute the shape and intensity of the lines measured in light scattering experiments.

KEY WORDS: Nonequilibrium stationary state; Rayleigh–Bénard system; fluctuations; correlation functions; fluctuating hydrodynamics; fast and slow variables; time scale perturbation theory; nonequilibrium hydrodynamic modes.

1. INTRODUCTION

Fluctuations in fluids away from thermal equilibrium have been of considerable interest since a number of years. As an example of a non-

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equilibrium fluid we will discuss here the Rayleigh–Bénard system: a simple fluid in a gravity field confined between two horizontal plates which are maintained at different temperatures so that the fluid is exposed to a stationary heat flux. We want to study the correlation functions in this system and discuss in particular situations where the boundary conditions and the variation of the macroscopic hydrodynamic quantities with position have to be taken into account.

Three methods have been applied in the literature to compute the correlation functions in the nonequilibrium stationary state: kinetic theory,[1] mode-coupling theory,[1–4] and fluctuating hydrodynamics.[5] The first two approaches start on a microscopic level to arrive at hydrodynamiclike equations for the correlation functions. In kinetic theory (which is only valid at low densities) this is achieved by the Chapman–Enskog method, while in mode-coupling theory projection operator techniques are employed. The third approach is more phenomenological. It assumes that the theory of fluctuating hydrodynamics as proposed by Landau and Lifshitz[6] to compute fluctuations around thermal equilibrium can be extended in a simple way to a nonequilibrium stationary state. Formal arguments to justify the application of fluctuating hydrodynamics to nonequilibrium stationary states have been given, based on mode-coupling theory,[7] on kinetic theory,[8] on a Fokker–Planck equation,[9] and on a master equation approach.[10]

For small temperature gradients one can use perturbation theories around equilibrium.[11–16] For large gradients the problem becomes more difficult, firstly because one can no longer use perturbation theory and secondly because one must take into account that the average quantities depend on position. The temperature gradient itself is not constant since the thermal conductivity depends on the local temperature. Kirkpatrick, Cohen, and Dorfman[17] were the first to compute the density–density correlation function for large gradients with kinetic and with mode-coupling theory. Measurements of the intensities of the Brillouin lines in the light-scattering spectrum are consistent with their results.[18]

In solving their equations Kirkpatrick *et al.* use the hydrodynamic modes from thermal equilibrium. Although this procedure is well suited for the linear regime, it requires infinite resummation methods for large temperature gradients. Clearly the use of different normal modes which are adapted to the actual nonequilibrium situation would make the calculation much more transparent. Several authors have already computed a subset of these nonequilibrium modes with no[19a] or with simplified boundary conditions[20–22] in order to discuss the central line in the light-scattering experiments or the convective instability.

In this series, which consists of three papers (hereafter referred to as I, II, III), we present a unified treatment of the correlation functions based on fluctuating hydrodynamics. Not only the density–density correlation function, relevant for light-scattering experiments, but all the hydrodynamic correlation functions will be discussed systematically. They are not only of interest for the transport theory far from equilibrium,[23–24] but, in addition, the velocity–velocity correlation function can be observed directly by laser–Doppler velocimetry. In our calculations we will use the nonequilibrium hydrodynamic modes which will be determined explicitly for realistic boundary conditions. While the theories mentioned above[19–22] are based on the Boussinesq approximation[25] of the hydrodynamic equations, and thus neglect the sound modes, we will discuss also these for the first time in a system where the spatial variation of the average quantities as well as the boundary conditions have to be taken into account explicitly. This solves the as yet open problem of the influence of the boundaries on the Brillouin lines in the presence of a large temperature gradient.[17] The case of a small temperature gradient, when a linear theory applies, has been discussed by Satten and Ronis[19b] and later in Ref. 12. These calculations are also consistent with the experiments of Ref. 18. A further discussion of the Brillouin lines in the presence of large gradients and boundaries is given in paper III. We remark that the nonequilibrium modes are interesting for their own sake, since they have a direct physical interpretation as collective phenomena. Moreover, they can be used to compute properties of the fluid other than correlation functions.

The first paper, which contains the general theory for arbitrary temperature profiles, has two main purposes: first, we demonstrate in the particular nonequilibrium system considered here that fluctuating hydrodynamics leads to the same equations for the correlation functions as mode-coupling theory and, for a dilute gas, kinetic theory do. This will supplement the more formal investigations in Refs. 7 and 8. Second, we derive the equations for the nonequilibrium modes by generalizing time-scaling methods from irreversible thermodynamics.[26] This technique is based on the fact that hydrodynamic processes evolve on two widely separated time scales: a fast time scale which is characterized by the speed of sound and a slow scale characterized by the speed of friction and heat diffusion processes or the speed of convective heating by flow perturbations in the presence of the macroscopic temperature gradient. The equations for the slow modes are identical to the linearized Boussinesq equations,[25] while those for the fast variables describe the sound modes.

In papers II and III we will then separately discuss the slow and fast processes and their contributions to the correlation functions taking into account the boundary conditions on the plates. In later publications we will also investigate fluids under stationary stress.

The plan of I is as follows: In Section 2 we describe the stationary state and the symmetry properties of the correlation functions. Then, in Section 3, we summarize the basic ideas of fluctuating hydrodynamics as they are relevant for our developments. In Section 4 we consider the equal-time correlation matrix and argue that it has a long-range part away from thermal equilibrium. We derive an equation from which this long-range part can be determined. In Sections 5 and 6 we discuss the hydrodynamic operator which is obtained by linearizing the hydrodynamic equations around the stationary state solution. We identify slow and fast variables and decouple these two sets by a time scale perturbation theory. These results allow us in Section 7 to derive the eigenvalue equations for the non-equilibrium hydrodynamic modes, which are the viscous, the viscoheat, and the sound modes. In Section 8 we compute formally the correlation matrix of the hydrodynamic variables in terms of these nonequilibrium modes. At the end we briefly summarize the main results.

2. CORRELATION FUNCTIONS

We consider a simple fluid in a homogeneous gravity field $\mathbf{g}$. The macroscopic behavior of the fluid is described by the nonlinear hydrodynamic equations.[27] In view of the boundary conditions to be applied it is most convenient to choose the pressure $p(\mathbf{r}, t)$, the temperature $T(\mathbf{r}, t)$, and the flow velocity $\mathbf{u}(\mathbf{r}, t)$ as the independent hydrodynamic variables. Then the equations read

$$\begin{aligned}
\frac{\partial p}{\partial t}+\mathbf{u}\cdot\nabla p &= -\frac{\gamma}{\chi_T}\nabla\cdot\mathbf{u}+\frac{\gamma-1}{\alpha T}(\boldsymbol{\tau}:\nabla\mathbf{u}-\nabla\cdot\mathbf{q})\\
\frac{\partial T}{\partial t}+\mathbf{u}\cdot\nabla T &= -\frac{\gamma-1}{\alpha}\nabla\cdot\mathbf{u}+\frac{1}{\rho c_V}(\boldsymbol{\tau}:\nabla\mathbf{u}-\nabla\cdot\mathbf{q})\\
\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u} &= -\frac{1}{\rho}\nabla p+\frac{1}{\rho}\nabla\cdot\boldsymbol{\tau}+\mathbf{g}
\end{aligned}\tag{2.1}$$

where ρ, α, χ_T, c_V and $\gamma=c_p/c_V$ are the mass density, the thermal expansion coefficient, the isothermal compressibility, the specific heat at constant volume per unit mass, and the ratio of the specific heats at constant pressure and constant volume, respectively. Furthermore $\boldsymbol{\tau}$ and $\mathbf{q}$ are the dissipative part of the stress tensor and the heat flux, respectively, given by the linear phenomenological laws

$$\begin{aligned}
\boldsymbol{\tau} &= 2\eta\overline{\nabla\mathbf{u}}+\zeta\nabla\cdot\mathbf{u}\mathsf{1}\\
\mathbf{q} &= -\lambda\nabla T
\end{aligned}\tag{2.2}$$

where $\overline{\nabla \mathbf{u}}$ is the symmetric and traceless part of $\nabla \mathbf{u}$, and η, ζ, λ are the shear viscosity, bulk viscosity, and thermal conductivity, respectively. The thermodynamic quantities ρ, α, χ_T, c_V, c_p, as well as the transport coefficients η, ζ, λ in Eqs. (2.1) and (2.2) all depend on $p(\mathbf{r}, t)$ and $T(\mathbf{r}, t)$ via the local equations of state.

The fluid is confined between two horizontal, infinite planes of distance d which have different temperatures T_1 and T_2, respectively. We assume that these temperatures are uniformly distributed on the plates and constant in time and that the fluid has reached a stationary state with vanishing flow velocity $\mathbf{u}$ (no convection). It is convenient to choose a Cartesian coordinate system such that gravity acts in the negative z direction, i.e., $\mathbf{g} = -g\mathbf{e}_z$, and the plates are located at $z = -d/2$ and $z = +d/2$. Then, according to (2.1) and (2.2), the pressure and temperature in the fluid have one-dimensional profiles, $p(z)$ and $T(z)$, which follow from the ordinary, nonlinear equations

$$\frac{dp}{dz} + g\rho = 0 \tag{2.3a}$$

$$\frac{d}{dz}\lambda\frac{dT}{dz} = 0 \tag{2.3b}$$

and the boundary conditions

$$\begin{gathered} p(d/2) = p_2 \\ T(-d/2) = T_1, \qquad T(d/2) = T_2 \end{gathered} \tag{2.4}$$

where p_2 is the outside pressure.

Assuming the macroscopic fields $p(z)$ and $T(z)$, together with $\mathbf{u} = 0$, are given, we address ourselves now to the spontaneous thermal fluctuations around the steady state solution. We restrict ourselves to the properties of the fluctuations on hydrodynamic scales only, i.e., the behavior averaged over lengths and times which are large compared to the mean free path and the mean free time between successive collisions of the fluid particles. The hydrodynamic length scale may (but need not) be much smaller than the macroscopic length scale which is of the order $L_\nabla \simeq [(1/a)(da/dz)]^{-1}$, where $a(z) = a[p(z), T(z)]$ stands for the macroscopic quantity which varies most with position. In order to study the fluctuations on hydrodynamic scales we describe the fluctuations phenomenologically by a stochastic process. Then the fluctuations in the independent hydrodynamic variables, i.e., the fluctuations in the pressure $\delta p(\mathbf{r}, t)$, in the temperature $\delta T(\mathbf{r}, t)$, and in the flow velocity $\delta \mathbf{u}(\mathbf{r}, t)$, are the basic stochastic variables. They are defined to be the deviations of the actual values of the pressure, the temperature, and the flow velocity at point $\mathbf{r}$ and time t (on the hydrodynamic scales) from their macroscopic values, i.e. $p(z)$, $T(z)$, and $\mathbf{u}=0$. For convenience we will use the vector notation

$$\delta\mathbf{a}=\begin{pmatrix}\delta p\\ \delta T\\ \delta\mathbf{u}\end{pmatrix} \tag{2.5}$$

referring to the components of $\delta\mathbf{a}$ as the p, T, and $\mathbf{u}$ components, respectively, and denoting them by Greek indices. Averages over the probability distribution functions underlying the stochastic process in the nonequilibrium steady state will be denoted by $\langle\cdots\rangle_{ss}$. Then, by definition,

$$\langle\delta\mathbf{a}(\mathbf{r}, t)\rangle_{ss}=0 \tag{2.6}$$

We wish to compute the correlation matrix which is defined as

$$\mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)=\langle\delta\mathbf{a}(\mathbf{r}_1, t_1)\,\delta\mathbf{a}(\mathbf{r}_2, t_2)\rangle_{ss} \tag{2.7}$$

From (2.7) follows immediately

$$\mathsf{M}(\mathbf{r}_2, t_2; \mathbf{r}_1, t_1)=\mathsf{M}^T(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \tag{2.8}$$

where M^T is the transposed matrix of M. Since the averages are stationary, as well as translationally and rotationally invariant in the x–y plane, it follows furthermore that the correlation matrix depends only on $t=t_1-t_2$, $\mathbf{r}=(\mathbf{1}-\mathbf{e}_z\mathbf{e}_z)\cdot(\mathbf{r}_1-\mathbf{r}_2)=(x_1-x_2, y_1-y_2, 0)$, z_1 and z_2 in the following manner[28]:

$$\begin{aligned}
M_{\alpha\beta}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)&=g_{\alpha\beta}(r_\parallel, z_1, z_2; t) && (\alpha, \beta=p, T)\\
\mathbf{M}_{\alpha\mathbf{u}}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)&=f_{\alpha1}\mathbf{e}_z+f_{\alpha2}\hat{\mathbf{r}}_\parallel && (\alpha=p, T)\\
\mathsf{M}_{\mathbf{uu}}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)&=h_1(\mathbf{1}-\mathbf{e}_z\mathbf{e}_z)+h_2\mathbf{e}_z\mathbf{e}_z+h_3\hat{\mathbf{r}}_\parallel\hat{\mathbf{r}}_\parallel\\
&\quad h_4\mathbf{e}_z\hat{\mathbf{r}}_\parallel+h_5\hat{\mathbf{r}}_\parallel\mathbf{e}_z
\end{aligned} \tag{2.9}$$

where $\hat{\mathbf{r}}_\parallel=\mathbf{r}_\parallel/r_\parallel$ and $g_{\alpha\beta}$, $f_{\alpha1}$, $f_{\alpha2}$, h_1,..., h_5 are scalar functions of $r_\parallel=|\mathbf{r}_\parallel|$, z_1, z_2, and t. Using (2.8) we obtain the symmetry relation

$$h_5(r_\parallel, z_1, z_2; t)=-h_4(r_\parallel, z_2, z_1; -t) \tag{2.10}$$

Hence to determine the complete hydrodynamic correlation matrix $\mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ for all times t_1, t_2 it is sufficient to compute the 11 scalar functions g_{pp}, g_{pT}, g_{TT}, f_{p1}, f_{p2}, f_{T1}, f_{T2}, h_1, h_2, h_3, and h_4 for times $t\geqslant 0$.

3. FLUCTUATING HYDRODYNAMICS

In order to compute the correlation matrix, i.e., the functions $g_{pp},..., h_4$, we will use the theory of fluctuating hydrodynamics[(6)] as generalized to fluctuations around a nonequilibrium stationary state.[(5)] The basic ideas as they are relevant for our later developments will be summarized now.

The equations for the stochastic process describing the fluctuations $\delta\mathbf{a}(\mathbf{r}, t)$ of the hydrodynamic variables consist of a systematic and a random part:

$$\frac{\partial}{\partial t}\delta\mathbf{a}(\mathbf{r}, t) = -\mathfrak{H}(\mathbf{r})\cdot\delta\mathbf{a}(\mathbf{r}, t) + \delta\mathbf{F}(\mathbf{r}, t) \tag{3.1}$$

One assumes that the systematic evolution is governed by the linear hydrodynamic operator $\mathfrak{H}(\mathbf{r})$ (acting on functions of $\mathbf{r}$) which is obtained from the nonlinear hydrodynamic equations (2.1), (2.2) by linearization around the stationary state solution $p(z)$, $T(z)$, given by Eqs. (2.3), (2.4), and $\mathbf{u} = 0$.[3] The explicit expression for $\mathfrak{H}(\mathbf{r})$ will be given in the next section. The random force term $\delta\mathbf{F}(\mathbf{r}, t)$ in (3.1) represents the overall effect due to the coupling of nonhydrodynamic degrees of freedom to the hydrodynamic fluctuations.

To compute the correlation matrix (2.7) one does not need to know all the details of the random forces, but only make a few assumptions concerning their stochastic properties. These are

$$\langle\delta\mathbf{F}(\mathbf{r}, t)\rangle_{ss} = 0 \tag{3.2a}$$

$$\langle\delta\mathbf{F}(\mathbf{r}_1, t_1)\,\delta\mathbf{F}(\mathbf{r}_2, t_2)\rangle_{ss} = \mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)\,\delta(t_1 - t_2) \tag{3.2b}$$

Equation (3.2a) holds by definition. (3.2b) express the basic postulate that $\delta\mathbf{F}$ is a stationary Markov process on the hydrodynamic time scale. This is physically quite reasonable because the nonhydrodynamic variables should have no memory on hydrodynamic scales. The covariance matrix $\mathbf{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)$ will be determined below.

In order to derive equations for the correlation matrix one first solves Eq. (3.1) formally for times $t_1 \geqslant t_2$. Suppressing space arguments the result is

$$\delta\mathbf{a}(t_1) = \mathfrak{U}(t_1 - t_2)\cdot\delta\mathbf{a}(t_2) + \int_{t_2}^{t_1} \mathfrak{U}(t_1 - \tau)\cdot\delta\mathbf{F}(\tau)\,d\tau \qquad (t_1 \geqslant t_2) \tag{3.3}$$

where

$$\mathfrak{U}(t) = \exp(-\mathfrak{H}t) \qquad (t \geqslant 0) \tag{3.4}$$

[3] Linearizing around the steady state is a reasonable procedure not too close to a critical point or a hydrodynamic instability, since then the fluctuations are small in amplitude.

is the time-evolution operator. Multiplying (3.3) from the right by $\delta\mathbf{a}(t_2)$ and averaging yields

$$\mathsf{M}(t_1, t_2) = \mathfrak{U}(t_1 - t_2) \cdot \mathsf{M}(t_2, t_2) \qquad (t_1 \geqslant t_2) \tag{3.5}$$

where (2.7) and (3.2a) have been used. In (3.5) the operator $\mathfrak{U}$ acts only on functions of $\mathbf{r}_1$. Furthermore we obtain from (3.3), (3.2b), (3.2c) and stationarity

$$\begin{aligned} M_{\alpha\beta}(t_2, t_2) = {} & \mathscr{U}_{\alpha\gamma}(t_1 - t_2)\, \mathscr{U}_{\beta\delta}(t_1 - t_2)\, M_{\gamma\delta}(t_2, t_2) \\ & + \int_{t_2}^{t_1} \mathscr{U}_{\alpha\gamma}(t_1 - \tau)\, \mathscr{U}_{\beta\delta}(t_1 - \tau)\, d\tau\, \Gamma_{\gamma\delta} \end{aligned} \tag{3.6}$$

where on the right-hand side (r.h.s.) the first $\mathfrak{U}$ acts on $\mathbf{r}_1$, while the second one acts on $\mathbf{r}_2$, and summation convent is used, as in the rest of the paper. Finally, differentiating (3.5) and (3.6) with respect to t_1 and setting in the latter equation $t_1 = t_2$, yields the following two equations for the unequal- and equal-time correlation matrices:

$$\frac{\partial}{\partial t_1} \mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = -\mathfrak{H}(\mathbf{r}_1) \cdot \mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \qquad (t_1 > t_2) \tag{3.7}$$

and

$$\mathscr{H}_{\alpha\gamma}(\mathbf{r}_1)\, M_{\gamma\beta}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2) + \mathscr{H}_{\beta\gamma}(\mathbf{r}_2)\, M_{\alpha\gamma}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2) = \Gamma_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \tag{3.8}$$

respectively. Solving Eqs. (3.7) and (3.8) one can find expressions for the correlation matrix M in terms of the still unknown covariance matrix $\mathbf{\Gamma}$ for all times $t_1 \geqslant t_2$, and, applying (2.8), also for $t_1 < t_2$.

The covariance matrix $\mathbf{\Gamma}(\mathbf{r}_1, t_2, \mathbf{r}_2)$ could be determined from (3.8) if the equal-time correlation matrix $\mathsf{M}(\mathbf{r}_1, \mathbf{r}_2, t_2)$ were known. However, this is only the case in thermal equilibrium. Therefore one proceeds as follows. First one computes the covariance matrix $\mathbf{\Gamma}_{\text{eq}}$ in thermal equilibrium by inserting in the left-hand side (l.h.s.) of (3.8) the hydrodynamic operator $\mathfrak{H}_{\text{eq}}$ and the equal-time correlation matrix M_{eq} from thermal equilibrium. Then one postulates that the nonequilibrium $\mathbf{\Gamma}$ can be obtained from $\mathbf{\Gamma}_{\text{eq}}$ by replacing all the equilibrium quantities (i.e., the thermodynamic and transport coefficients) appearing in $\mathbf{\Gamma}_{\text{eq}}$ as parameters by their position-dependent steady state values. By this postulate one archieves[29] (i) that the correlation lengths of the random forces are of microscopic order (like in equilibrium) and (ii) that the random forces in a point $\mathbf{r}_1$ behave statistically as if they were embedded in an equilibrium environment characterized by the local values of the steady state quantities. The property (ii) is consistent with the fact that the macroscopic fields effectively do not change over microscopic distances, i.e., over the correlation lengths of the random forces.

To compute $\boldsymbol{\Gamma}$ we therefore start from the equal-time correlation matrix M_{eq} in thermal equilibrium. Here the correlation length is of microscopic order (assuming we are away from a critical point), i.e., on hydrodynamic length scales M_{eq} is short-range,

$$\mathsf{M}_{\mathrm{eq}}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2) \equiv \mathsf{A}(\mathbf{r}_1, \mathbf{r}_2) = \mathsf{A}^{(0)}\, \delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.9}$$

with[30],[31]

$$\mathsf{A}^{(0)} = k_B T \begin{pmatrix} \dfrac{\gamma}{\chi_T} & \dfrac{\gamma-1}{\alpha} & 0 \\ \dfrac{\gamma-1}{\alpha} & \dfrac{T}{\rho c_V} & 0 \\ 0 & 0 & \dfrac{1}{\rho}\mathbf{1} \end{pmatrix} \tag{3.10}$$

where k_B is the Boltzmann constant. Using now $\mathfrak{H}_{\mathrm{eq}}$ [the explicit expression of which can easily be obtained as a special case of Eqs. (4.5)–(4.8) below] one can straightforwardly compute $\boldsymbol{\Gamma}_{\mathrm{eq}}$ from (3.8)–(3.10).

It turns out that all matrix elements of $\boldsymbol{\Gamma}_{\mathrm{eq}}(\mathbf{r}_1, \mathbf{r}_2)$ are proportional to $\nabla_1 \nabla_1\, \delta(\mathbf{r}_1 - \mathbf{r}_2)$. In order to replace the equilibrium parameters in $\boldsymbol{\Gamma}_{\mathrm{eq}}$ by their corresponding position-dependent steady state quantities it is therefore convenient to determine first from $\boldsymbol{\Gamma}_{\mathrm{eq}}$ the correlation properties of the random heat flux $\delta\mathbf{Q}$ and the random stress tensor $\delta\mathsf{T}$ which are defined by [4]

$$\delta\mathbf{F} = \begin{pmatrix} -\dfrac{\gamma-1}{\alpha T} \nabla \cdot \delta\mathbf{Q} \\ -\dfrac{1}{\rho c_V} \nabla \cdot \delta\mathbf{Q} \\ \dfrac{1}{\rho} \nabla \cdot \delta\mathsf{T} \end{pmatrix} \tag{3.11}$$

One thus finds the Landau–Lifshitz expressions[6]

$$\begin{aligned} \langle \delta Q_i(\mathbf{r}_1, t_1)\, \delta Q_j(\mathbf{r}_2, t_2) \rangle_{\mathrm{eq}} &= 2k_B T^2 \lambda \delta_{ij} \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(t_1 - t_2) \\ \langle \delta Q_i(\mathbf{r}_1, t_1)\, \delta T_{kl}(\mathbf{r}_2, t_2) \rangle_{\mathrm{eq}} &= 0 \\ \langle \delta T_{ij}(\mathbf{r}_1, t_1)\, \delta T_{kl}(\mathbf{r}_2, t_2) \rangle_{\mathrm{eq}} &= 2k_B T[\eta(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl}) \\ &\quad + \zeta \delta_{ij}\delta_{kl}]\, \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(t_1 - t_2) \end{aligned} \tag{3.12}$$

[4] That $\delta\mathbf{F}$ can be written in this form follows from the conservation laws of mass, momentum, and energy in the fluid.[6]

where latin indices denote Cartesian vector components. It is these relations (3.12) which are now generalized to the nonequilibrium case by postulating that the values of the temperature T and the transport coefficients η, ζ, λ have to be taken in the point $\mathbf{r}_1$.

Finally, in order to find the nonequilibrium covariance matrix $\boldsymbol{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)$, one inserts (3.11) into (3.2b) and uses the local version of (3.12). The result appears in its most symmetrical form when one transforms from $\mathbf{r}_1$ and $\mathbf{r}_2$ to the center of mass and the relative coordinates

$$\mathbf{R} = \tfrac{1}{2}(\mathbf{r}_1 + \mathbf{r}_2), \qquad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \tag{3.13}$$

Then one finds $\boldsymbol{\Gamma}$ as a sum of three terms

$$\boldsymbol{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) = \boldsymbol{\Gamma}^{(0)}_{ij}(R_z)\, \partial^2_{ij}\delta(\mathbf{r}) + \boldsymbol{\Gamma}^{(1)}_i(R_z)\, \partial_i \delta(\mathbf{r}) + \boldsymbol{\Gamma}^{(2)}(R_z)\, \delta(\mathbf{r}) \tag{3.14}$$

where the first term on the r.h.s. is identical to the equilibrium expression, the average quantities being taken in R_z, while $\boldsymbol{\Gamma}^{(1)}_i$ and $\boldsymbol{\Gamma}^{(2)}$ are corrections of first resp. second order in the gradient $\partial/\partial R_z$.

Although explicit expressions for $\boldsymbol{\Gamma}^{(0)}_{ij}$, $\boldsymbol{\Gamma}^{(1)}_i$, and $\boldsymbol{\Gamma}^{(2)}$ in terms of the thermodynamic and transport quantities can be given, we will not do so here because, as we will see later, $\boldsymbol{\Gamma}^{(0)}_{ij}$ will cancel and $\boldsymbol{\Gamma}^{(1)}_i$ and $\boldsymbol{\Gamma}^{(2)}$ can be neglected.

We will now go back to Eq. (3.8) and discuss the equal-time correlation matrix $\mathsf{M}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2)$.

4. EQUAL-TIME CORRELATION MATRIX

Instead of solving Eq. (3.8) directly for the equal-time correlation matrix it is convenient to split $\mathsf{M}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2)$ into two parts:

$$\mathsf{M}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2) = \mathsf{A}(\mathbf{r}_1, \mathbf{r}_2) + \mathsf{D}(\mathbf{r}_1, \mathbf{r}_2) \tag{4.1}$$

where

$$\mathsf{A}(\mathbf{r}_1, \mathbf{r}_2) = \mathsf{A}^{(0)}(\mathbf{r}_1)\, \delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{4.2}$$

is the short-range, local equilibrium correlation matrix, while $\mathsf{D}(\mathbf{r}_1, \mathbf{r}_2)$ vanishes in equilibrium, and, as we will see below, is long range. $\mathsf{A}^{(0)}(\mathbf{r}_1)$ is given by (3.10) with the prescription that the values of the equilibrium quantities $T, \rho, \alpha, \chi_T, c_V, \gamma$ have to be replaced by their corresponding steady state quantities taken in the point $\mathbf{r}_1$. Owing to stationarity, $\mathsf{M}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2)$ and, hence, $D(\mathbf{r}_1, \mathbf{r}_2)$ do not depend on t_2. Inserting the ansatz (4.1) into (3.8) we obtain for $\mathsf{D}(\mathbf{r}_1, \mathbf{r}_2)$ the equation

$$\mathscr{H}_{\alpha\beta}(\mathbf{r}_1)\, D_{\gamma\beta}(\mathbf{r}_1, \mathbf{r}_2) + \mathscr{H}_{\beta\gamma}(\mathbf{r}_2)\, D_{\alpha\gamma}(\mathbf{r}_1, \mathbf{r}_2) = -B_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \tag{4.3}$$

where

$$B_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = \mathscr{H}_{\alpha\gamma}(\mathbf{r}_1)\, A_{\gamma\beta}(\mathbf{r}_1, \mathbf{r}_2) + \mathscr{H}_{\beta\gamma}(\mathbf{r}_2)\, A_{\alpha\gamma}(\mathbf{r}_1, \mathbf{r}_2) - \Gamma_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) \quad (4.4)$$

To compute $\mathsf{B}(\mathbf{r}_1, \mathbf{r}_2)$ we split the hydrodynamic operator $\mathscr{H}(\mathbf{r})$ into an Euler– and a Navier–Stokes part which can be deduced from linearizing the reversible and the dissipative parts, respectively, of the nonlinear hydrodynamic equations (2.1), (2.2) around the stationary state solution:

$$\mathfrak{H}(\mathbf{r}) = \mathfrak{E}(\mathbf{r}) + \mathfrak{N}(\mathbf{r}) \quad (4.5)$$

The Euler operator reads explicitly

$$\mathfrak{E}(\mathbf{r}) = \begin{pmatrix} 0 & 0 & \dfrac{\gamma}{\chi_T}\nabla - g\rho\mathbf{e}_z \\ 0 & 0 & \dfrac{\gamma - 1}{\alpha}\nabla + \dfrac{dT}{dz}\mathbf{e}_z \\ \dfrac{1}{\rho}\nabla + g\chi_T\mathbf{e}_z & -g\alpha\mathbf{e}_z & 0 \end{pmatrix} \quad (4.6)$$

while, for brevity, we give the Navier–Stokes operator in the form

$$\mathfrak{N}(\mathbf{r}) \cdot \delta\mathbf{a} = \begin{pmatrix} \dfrac{\gamma - 1}{\alpha T}\nabla \cdot \delta\mathbf{q} \\ \dfrac{1}{\rho c_V}\nabla \cdot \delta\mathbf{q} \\ -\dfrac{1}{\rho}\nabla \cdot \delta\boldsymbol{\tau} \end{pmatrix} \quad (4.7)$$

with

$$\delta\mathbf{q} = -\lambda\nabla\delta T - \left[\left(\frac{\partial\lambda}{\partial p}\right)_T \delta p + \left(\frac{\partial\lambda}{\partial T}\right)_p \delta T\right]\frac{dT}{dz}\mathbf{e}_z \quad (4.8a)$$

$$\delta\boldsymbol{\tau} = 2\eta\,\overline{\nabla\delta\mathbf{u}} + \zeta\nabla \cdot \delta\mathbf{u}\,\mathbf{1} \quad (4.8b)$$

In (4.6)–(4.8) the values of the steady state quantities ρ, α, χ_T, c_V, γ, dT/dz, λ, η, ζ, $(\partial\lambda/\partial p)_T$ and $(\partial\lambda/\partial T)_p$ have to be taken in the point $\mathbf{r}$. After inserting of (4.5) into (4.4), $\mathsf{B}(\mathbf{r}_1, \mathbf{r}_2)$ consists of three contributions

$$\mathsf{B} = \mathsf{C} + \boldsymbol{\Delta} - \boldsymbol{\Gamma} \quad (4.9)$$

which are given by

$$C_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = \mathscr{E}_{\alpha\gamma}(\mathbf{r}_1)\, A_{\gamma\beta} + \mathscr{E}_{\beta\gamma}(\mathbf{r}_2)\, A_{\alpha\gamma} \tag{4.10}$$

$$\Delta_{\alpha\beta}(\mathbf{r}_1, \mathbf{r}_2) = \mathscr{N}_{\alpha\gamma}(\mathbf{r}_1)\, A_{\gamma\beta} + \mathscr{N}_{\beta\gamma}(\mathbf{r}_2)\, A_{\alpha\gamma} \tag{4.11}$$

and (3.14).

In calculating $\mathsf{C}(\mathbf{r}_1, \mathbf{r}_2)$ we find that all the gravity terms cancel. Nonvanishing contributions arise for two reasons: (i) because of the term $(dT/dz)\,\mathbf{e}_z$ in $\mathfrak{E}$ which originates from the nonlinear convection term $\mathbf{u}\cdot\nabla T$ in Eqs. (2.1) and (ii) because the local equilibrium matrix $\mathsf{A}(\mathbf{r}_1, \mathbf{r}_2)$ does not depend only on $\mathbf{r}_1 - \mathbf{r}_2$, as in equilibrium, but on both coordinates separately. We obtain from (4.10), (4.2), and (3.10)

$$\mathsf{C}(\mathbf{r}_1, \mathbf{r}_2) = \mathsf{C}^{(0)}(\mathbf{r}_1)\, \delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{4.12}$$

with

$$\mathsf{C}^{(0)} = k_B \frac{dT}{dz} \begin{bmatrix} 0 & 0 & \dfrac{\gamma}{\rho\chi_T}\mathbf{e}_z \\ 0 & 0 & \dfrac{\alpha T + \gamma - 1}{\rho\alpha}\mathbf{e}_z \\ \dfrac{\gamma}{\rho\chi_T}\mathbf{e}_z & \dfrac{\alpha T + \gamma - 1}{\rho\alpha}\mathbf{e}_z & 0 \end{bmatrix} \tag{4.13}$$

$\boldsymbol{\Delta}(\mathbf{r}_1, \mathbf{r}_2)$ can be computed straightforwardly. Expressing the result in the center of mass and relative coordinates [Eq. (3.13)] one obtains an expression of the form

$$\boldsymbol{\Delta}(\mathbf{r}_1, \mathbf{r}_2) = \boldsymbol{\Delta}_{ij}^{(0)}(R_z)\, \partial_{ij}^2 \delta(\mathbf{r}) + \boldsymbol{\Delta}_i^{(1)}(R_z)\, \partial_i \delta(\mathbf{r}) + \boldsymbol{\Delta}^{(2)}(R_z)\, \delta(\mathbf{r}) \tag{4.14}$$

where the terms on the r.h.s. are of zeroth, first, and second order in the gradient $\partial/\partial R_z$, respectively. Since we will not need explicit expressions for $\boldsymbol{\Delta}_{ij}^{(0)}$, $\boldsymbol{\Delta}_i^{(1)}$, and $\boldsymbol{\Delta}^{(2)}$ we will not give them here. The zeroth-order term is just the equilibrium expression, the average quantities being taken in R_z. Hence it follows from (4.9), (3.14), and (4.14)

$$\boldsymbol{\Delta}_{ij}^{(0)}(R_z) = \Gamma_{ij}^{(0)}(R_z) \tag{4.15}$$

since in equilibrium B, C and all gradients with respect to R_z vanish.

Using the Eqs. (3.14), (4.14), and (4.15) in (4.9) we find

$$\mathsf{B}(\mathbf{r}_1, \mathbf{r}_2) = \mathsf{C}(\mathbf{r}_1, \mathbf{r}_2) + \boldsymbol{\Lambda}_i^{(1)}(R_z)\, \partial_i \delta(\mathbf{r}) + \boldsymbol{\Lambda}^{(2)}(R_z)\, \delta(\mathbf{r}) \tag{4.16}$$

where C is given by (4.12), (4.13) and $\boldsymbol{\Lambda}_i^{(1)} = \boldsymbol{\Delta}_i^{(1)} - \boldsymbol{\Gamma}_i^{(1)}$, $\boldsymbol{\Lambda}^{(2)} = \boldsymbol{\Delta}^{(2)} - \boldsymbol{\Gamma}^{(2)}$. The first two terms on the r.h.s. of (4.16) are of first order in $\partial/\partial R_z$, the third one is of second order. Inserting the expression (4.16) into (4.3), one has now an equation from which $\mathsf{D}(\mathbf{r}_1, \mathbf{r}_2)$ can be computed.

We shall end this section with a few remarks:

1. At this stage we can already argue that the solution $\mathsf{D}(\mathbf{r}_1, \mathbf{r}_2)$ of Eq. (4.3) is a long-range function of the relative coordinate $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. Expressing, namely, the l.h.s. of (4.3) in terms of $\mathbf{r}$ and $\mathbf{R}$ yields a second-order differential equation in $\mathbf{r}$ space. The inhomogeneity B, on the other hand, contains only terms proportional to $\delta(\mathbf{r})$ and $\nabla\delta(\mathbf{r})$. Equation (4.3) is therefore similar to the Poisson equation $\nabla^2\phi = 4\pi\delta(\mathbf{r})$. No short-range ϕ or D, i.e., no functions proportional to $\delta(\mathbf{r})$, or derivatives thereof, can be special solutions of such equations. Finally, the homogeneous part of (4.3) has no short-range solutions either, since it has no nontrivial solutions, because $\mathfrak{H}$ is a strictly positive operator.

2. Long-range correlations at equal times do not appear as a consequence of the inhomogeneity of the macroscopic state alone. For example, turning off the heat flux, but keeping the gravity field yields a state of inhomogeneous equilibrium with $dp/dz \neq 0$ but with zero entropy production.[5] Then B vanishes, so that still $\mathsf{D} = 0$. Therefore long-range correlations at equal time require a true nonequilibrium macroscopic state.

3. The equation (4.3) for D, derived here from fluctuating hydrodynamics, is identical with that derived in Ref. 1 from mode-coupling theory in the absence of the gravity field (when the so-called mode-coupling amplitudes are computed explicitly with the aid of Ref. 30). This follows form the fact that the terms $\Lambda_i^{(1)}$ and $\Lambda^{(2)}$, which are proportional to the transport coefficients, can be neglected when compared to $\mathsf{C}(\mathbf{r}_1, \mathbf{r}_2)$, as we will see in Section 8.

5. HYDRODYNAMIC OPERATOR

In order to solve the equation (3.7) for the unequal-time correlation matrix and the equation (4.3) for the long-range part of equal-time correlation matrix we will use a spectral decomposition of the non-equilibrium hydrodynamic operator $\mathfrak{H}(\mathbf{r})$. Since $\mathfrak{H}$ is not a symmetric operator one must distinguish a right- and left-eigenvalue problem:

$$\mathfrak{H} \cdot \mathbf{a}_K^R(\mathbf{r}) = s_K \mathbf{a}_K^R(\mathbf{r}) \tag{5.1a}$$

$$\mathfrak{H}^\dagger \cdot \mathbf{a}_K^L(\mathbf{r}) = s_K^* \mathbf{a}_K^L(\mathbf{r}) \tag{5.1b}$$

[5] A nonvanishing heat flux $\mathbf{q} = -\lambda(dT/dz)\,\mathbf{e}_z$ [cf. Eq. (2.2)] forced upon the system by the boundary conditions (2.4) gives rise to a local entropy production[27] $\sigma = (\lambda/T^2)(dT/dz)^2$ in the interior of the fluid. The pressure gradient caused by gravity does not produce entropy.

Here $\mathfrak{H}^\dagger$ is the adjoint operator of $\mathfrak{H}$ [in the scalar product to be defined in (5.2) below], K is a multi-index labelling the eigenmodes, s_K is an eigenvalue, s_K^* its complex conjugate, and $\mathbf{a}_K^R(\mathbf{r})$, $\mathbf{a}_K^L(\mathbf{r})$ denote the right and left eigenvectors, respectively. We assume that the right and left eigenvectors form a complete biorthogonal system, i.e.,

$$(\mathbf{a}_K^L, \mathbf{a}_{K'}^R) \equiv \int \mathbf{a}_K^{L*}(\mathbf{r}) \cdot \mathbf{a}_{K'}^R(\mathbf{r})\, d\mathbf{r} = \delta_{KK'} \tag{5.2}$$

$$\sum_K \mathbf{a}_K^R(\mathbf{r})\, a_K^{L*}(\mathbf{r}') = \mathbf{1}\, \delta(\mathbf{r} - \mathbf{r}') \tag{5.3}$$

Since we are not able to solve the eigenvalue problem of $\mathfrak{H}$ exactly, it is the aim of this and the following section to provide a method for an approximate, partial diagonalization of $\mathfrak{H}$. This will be based on a separation of fast and slow time scales.

Consider thereto the initial-value problem

$$\frac{\partial}{\partial t} \hat{\mathbf{a}}(\mathbf{r}, t) = -\mathfrak{H}(\mathbf{r}) \cdot \hat{\mathbf{a}}(\mathbf{r}, t) \qquad (t > 0) \tag{5.4}$$

with

$$\hat{\mathbf{a}}(\mathbf{r}, 0) = \hat{\mathbf{a}}_0(\mathbf{r}) \tag{5.5}$$

$\mathbf{a}_0(\mathbf{r})$ can be interpreted as a given small perturbation of the hydrodynamic fields from their stationary state values $p(z)$, $T(z)$, $\mathbf{u} = 0$, caused by some external forces. Equation (5.4) describes then how this perturbation decays to zero when the external forces are turned off at time $t = 0$.

The solution of the initial-value problem is complicated by the fact that the average quantities occurring in the operator $\mathfrak{H}$ [cf. Eqs. (4.5)–(4.8)] depend on the z coordinate. Thus, in principle, Eq. (5.4) has no plane wave solutions in the z direction. However, for the applications we have in mind, we restrict ourselves here to those modes which can be described approximately by plane waves within horizontal fluid layers of thickness $l_0 \ll L_\nabla$, where l_0 is chosen such that the spatial variation of the average quantities can be neglected within each layer.[6] The length scale on which the average quantities do vary is given by $L_\nabla \simeq [(1/a)(da/dz)]^{-1}$, where $a(z) = a(p(z), T(z))$ stands for the average quantity which varies most with position. We will refer to L_∇ as the macroscopic length scale.

[6] In a usual Rayleigh–Bénard cell one has $d \ll L_\nabla$ and can therefore set $l_0 = d$, i.e., one may neglect the spatial variation of the average quantities throughout the whole system, at least as far as the slow modes are concerned.[25]

Therefore we can choose on the z axis a reference point R_z in the fluid and consider a horizontal fluid layer V_{R_z} of thickness $l_0 \ll L_\nabla$. Neglecting at first the boundary conditions we assume that for points $\mathbf{r} \in V_{R_z}$ the initial perturbation can be described by a plane wave

$$\hat{\mathbf{a}}_0(\mathbf{r}) \approx \mathbf{a}_{0\mathbf{k}}(R_z)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \qquad (\mathbf{r} \in V_{R_z}) \tag{5.6}$$

with a wave vector $\mathbf{k} = \mathbf{k}(R_z)$ such that

$$L_\nabla k_z \gg l_0 k_z \gtrsim 1 \tag{5.7}$$

Then $\mathbf{k}$ is a "good quantum number" within the layer V_{R_z}, so that, following the dynamics (5.4) for an infinitesimal time t, $\hat{\mathbf{a}}_0(\mathbf{r})$ will evolve into a state $\hat{\mathbf{a}}(\mathbf{r}, t)$ that can be described within V_{R_z} as a wave with the same wave vector

$$\hat{\mathbf{a}}(\mathbf{r}, t) \approx \mathbf{a}_{\mathbf{k}}(R_z, t)\, e^{-i\mathbf{k}\cdot\mathbf{r}} \qquad (\mathbf{r} \in V_{R_z}) \tag{5.8}$$

From the 5×5 system (5.4) follows that the amplitude $\mathbf{a}_k(R_z, t)$ can be expressed as a sum of five modes, all with wave vector $\mathbf{k}$, that decay to zero according to certain, yet unknown, "local" eigenvalues $s_{kj}(R_z)\,(j = 1,..., 5)$:

$$\mathbf{a}_{\mathbf{k}}(R_z, t) = \sum_{j=1}^{5} \mathbf{A}_{kj}(R_z)\, e^{-s_{\mathbf{k}j}(R_z)t} \tag{5.9}$$

The real and imaginary part of s_{kj} are the damping constant and the frequency of the jth mode, respectively. $|\mathrm{Im}\, s_{kj}|/k$ is its "local" phase velocity. The mean-free path, defined as

$$l_j(\mathbf{k}) = \frac{|\mathrm{Im}\, s_{kj}|}{k\, \mathrm{Re}\, s_{kj}} \tag{5.10}$$

is a measure for the distance the wave can propagate before it is effectively damped to zero. If now $\mathbf{k}$ is chosen such that

$$l_j(\mathbf{k}) \cdot \frac{k_z}{k} \lesssim l_0 \ll L_\nabla \tag{5.11}$$

then points on the wave fronts of (5.8) will not be able to leave the layer V_{R_z} during their lifetime and, conversely, no waves from a neighboring layer will penetrate into V_{R_z}. As we will see, there are sound modes moving in the z direction that violate condition (5.11). We will come back to this later. For simplicity we assume first that (5.11) holds. Then the ansatz (5.8) describes the time evolution of (5.6) within V_{R_z} for all times $t > 0$.

Inserting thus (5.8) into (5.4) and using that the average quantities are approximately constant in V_{R_z}, we obtain the dynamical equation

$$\frac{\partial}{\partial t}\mathbf{a}_{\mathbf{k}}(R_z, t) = -\mathsf{H}_{R_z}(\mathbf{k})\cdot\mathbf{a}_{\mathbf{k}}(R_z, t) \qquad (t>0) \tag{5.12}$$

where the hydrodynamic matrix $\mathsf{H}_{R_z}(\mathbf{k})$ is found with the aid of (4.5)–(4.8) to read[7]

$$\mathsf{H}_{R_z}(\mathbf{k}) = \begin{bmatrix} 0 & \dfrac{\alpha\gamma}{\chi_T}D_T k^2 & i\dfrac{\gamma}{\chi_T}\mathbf{k} - g\rho\mathbf{e}_z \\ 0 & \gamma D_T k^2 & i\dfrac{\gamma-1}{\alpha}\mathbf{k} + \dfrac{dT}{dz}\mathbf{e}_z \\ i\dfrac{1}{\rho}\mathbf{k} + g\chi_T\mathbf{e}_z & -g\alpha\mathbf{e}_z & \nu k^2\,\mathbf{1} + (\Gamma_l - \nu)\,\mathbf{kk} \end{bmatrix} \tag{5.13}$$

with the prescription that the values of the steady state quantities have to be taken in the reference point R_z. In (5.13) we have used the thermodynamic identity $\alpha^2 T/\rho\chi_T = c_p - c_V$, and we have introduced the kinematic viscosity ν, the longitudinal viscosity Γ_l, and the thermal diffusivity D_T which are defined as

$$\nu = \frac{\eta}{\rho}, \qquad \Gamma_l = \frac{4}{3}\nu + \frac{\zeta}{\rho}, \qquad D_T = \frac{\lambda}{\rho c_p} \tag{5.14}$$

respectively.

In order to solve (5.12), or, equivalently, to determine the five "local" eigenvalues $s_{kj}(R_z)$ $(j = 1,..., 5)$, of $\mathsf{H}_{R_z}(\mathbf{k})$ it is not convenient to use the Cartesian components of the flow velocity $\mathbf{u}_{\mathbf{k}}$ as variables. Instead we express $\mathbf{u}_{\mathbf{k}}$ in terms of the longitudinal potential $\phi_{\mathbf{k}}$ and the z components $v_{\mathbf{k}}$ and $\xi_{\mathbf{k}}$ of the transversal velocity and the vorticity, respectively, i.e., we use

$$\mathbf{u}_{\mathbf{k}} = i\phi_{\mathbf{k}}\mathbf{k} - v_{\mathbf{k}}\frac{\mathbf{k}\times(\mathbf{k}\times\mathbf{e}_z)}{k_\parallel^2} + i\xi_{\mathbf{k}}\frac{\mathbf{k}\times\mathbf{e}_z}{k_\parallel^2} \tag{5.15}$$

where

$$k_\parallel^2 = k_x^2 + k_y^2$$

and

$$\phi_{\mathbf{k}} = -i\frac{1}{k^2}\mathbf{k}\cdot\mathbf{u}_{\mathbf{k}} \tag{5.16a}$$

$$v_{\mathbf{k}} = \mathbf{e}_z\cdot\left(\mathbf{1} - \frac{\mathbf{kk}}{k^2}\right)\cdot\mathbf{u}_{\mathbf{k}} \tag{5.16b}$$

$$\xi_{\mathbf{k}} = i\mathbf{e}_z\cdot(\mathbf{k}\times\mathbf{u}_{\mathbf{k}}) \tag{5.16c}$$

[7] We assume that the second term in (4.8a) is small compared to the first term and can thus be neglected.

Multiplying the **u** component of (5.12) by $\mathbf{e}_z \cdot (\mathbf{k} \times)$ one finds that the vorticity is decoupled from the other variables:

$$\frac{\partial}{\partial t} \xi_{\mathbf{k}} = -\nu k^2 \xi_k \tag{5.17}$$

Hence, one eigenvalue of $\mathfrak{H}_{R_z}(\mathbf{k})$ is

$$s_{\mathbf{k}1} = \nu k^2 \tag{5.18}$$

The corresponding mode is called the viscous mode.

The remaining four variables $p_{\mathbf{k}}$, $T_{\mathbf{k}}$, $\phi_{\mathbf{k}}$, and $v_{\mathbf{k}}$, that determine the eigenvalues $s_{\mathbf{k}2}, \ldots, s_{\mathbf{k}5}$ are all coupled. To be able to compare the relative magnitudes of the various elements of $\mathfrak{H}(\mathbf{k})$ we rescale the variables in such a way that they all have the same dimension. The most natural way of doing so is to divide each variable by the square root of its local equilibrium correlation strength as given by (3.10). Thus we set

$$\begin{aligned} \bar{p}_{\mathbf{k}} &= \left(\frac{\chi_T}{\gamma}\right)^{1/2} p_{\mathbf{k}}, \qquad & \bar{T}_k &= \left(\frac{\rho c_V}{T}\right)^{1/2} T_{\mathbf{k}} \\ \bar{\phi}_k &= \sqrt{\rho}\, k \phi_{\mathbf{k}}, \qquad & \bar{v}_{\mathbf{k}} &= \sqrt{\rho}\, v_{\mathbf{k}} \end{aligned} \tag{5.19}$$

In the following we will always denote scaled quantities with bars. In terms of the scaled variables the dynamical equations can be written in the form

$$\frac{\partial}{\partial t} \begin{pmatrix} \bar{\mathbf{x}}_{\mathbf{k}} \\ \bar{\mathbf{y}}_k \end{pmatrix} = -\begin{pmatrix} \bar{\mathsf{H}}_{xx} & \bar{\mathsf{H}}_{xy} \\ \bar{\mathsf{H}}_{yx} & \bar{\mathsf{H}}_{yy} \end{pmatrix} \cdot \begin{pmatrix} \bar{\mathbf{x}}_k \\ \bar{\mathbf{y}}_k \end{pmatrix} \tag{5.20}$$

where we have combined the temperature and the transversal velocity on the one hand, and the pressure and the longitudinal potential on the other hand,

$$\bar{\mathbf{x}}_{\mathbf{k}} = \begin{pmatrix} \bar{T}_{\mathbf{k}} \\ \bar{v}_{\mathbf{k}} \end{pmatrix}, \qquad \bar{\mathbf{y}}_{\mathbf{k}} = \begin{pmatrix} \bar{p}_{\mathbf{k}} \\ \bar{\phi}_{\mathbf{k}} \end{pmatrix} \tag{5.21}$$

into separate subvectors for reasons which will become clear below. The elements of the scaled dynamical matrix are

$$\bar{\mathsf{H}}_{xx} = \begin{pmatrix} \gamma D_T k^2 & \left(\dfrac{c_V}{T}\right)^{1/2} \dfrac{dT}{dz} \\ -g\alpha \left(\dfrac{T}{c_V}\right)^{1/2} \dfrac{k_\parallel^2}{k^2} & \nu k^2 \end{pmatrix}$$

$$\bar{\mathsf{H}}_{xy} = \begin{pmatrix} 0 & -\left(\dfrac{\gamma-1}{\gamma}\right)^{1/2} ck + i\left(\dfrac{c_V}{T}\right)^{1/2} \dfrac{dT}{dz}\dfrac{k_z}{k} \\ g\dfrac{\gamma}{c}\dfrac{k_\parallel^2}{k^2} & 0 \end{pmatrix}$$

$$\bar{\mathsf{H}}_{yx} = \begin{pmatrix} [\gamma(\gamma-1)]^{1/2} D_T k^2 & -\dfrac{g}{c} \\ ig\alpha\left(\dfrac{T}{c_V}\right)^{1/2} \dfrac{k_z}{k} & 0 \end{pmatrix}$$

$$\bar{\mathsf{H}}_{yy} = \begin{pmatrix} 0 & -ck\left(1 + i\dfrac{g}{c^2}\dfrac{k_z}{k^2}\right) \\ ck\left(1 - i\dfrac{g}{c^2}\dfrac{k_z}{k^2}\right) & \Gamma_l k^2 \end{pmatrix} \tag{5.22}$$

where

$$c = \left(\frac{\gamma}{\rho\chi_T}\right)^{1/2} \tag{5.23}$$

is the local speed of sound.

We will now identify fast and slow variables by estimating the various elements of $\bar{\mathsf{H}}$. Assuming that k lies in the regime

$$k_1 \ll k \ll k_2 \tag{5.24}$$

with $k_1 \approx (1/c)(c_V/T)^{1/2}\,(dT/dz)$ and $k_2 \approx c/D_T$ one observes that the terms proportional to ck (which appear only in $\bar{\mathsf{H}}_{xy}$ and $\bar{\mathsf{H}}_{yy}$) are much larger than all the other elements.[8] These terms originate from the equilibrium part of the Euler operator (4.6). The upper bound k_2 ensures that the Navier–Stokes terms which are of the order $D_T k^2$ are small compared to ck. Wave vectors of the order k_2 probe spatial distances of the order of 10 Å which are too small to be treated hydrodynamically. The lower bound k_1 ensures that the gradient term, $(c_V/T)^{1/2}\,(dT/dz)$, be much smaller than ck. k_1 is typically of the order $1/L_\nabla$. Hence, the bounds (5.24) are no new restrictions on our theory, since they are consistent with (5.7) and with the limitations of a hydrodynamic treatment in general.

Neglecting all terms that are much smaller than ck for a first estimate, one finds that the eigenvalues of $\bar{\mathsf{H}}$ are either of the order zero or ck. This means that the dynamic processes described by Eq. (5.12) evolve on two

[8] If we choose water under normal conditions ($T \approx 300\,\mathrm{K}$, $p \approx 1\,\mathrm{atm}$) as an example of a typical liquid, we have[32]: $\gamma \approx 1$, $D_T \approx \nu \approx \Gamma_l \approx 10^{-2}\,\mathrm{cm^2\,sec^{-1}}$, $c \approx 10^5\,\mathrm{cm\,sec^{-1}}$, $\alpha \approx 3\times10^{-4}\,\mathrm{K^{-1}}$, $c_V \approx 4\times10^7\,\mathrm{cm^2\,sec^{-2}\,K^{-1}}$. Furthermore $g = 10^3\,\mathrm{cm\,sec^{-2}}$.

widely separated time scales. $\tau_{fa}=1/ck$ is the characteristic unit on the fast time scale.

If we follow the dynamics (5.12) for a time of order τ_{fa}, then the influence of $\bar{\mathsf{H}}_{yy}$ on the time evolution is of order unity, while $\bar{\mathsf{H}}_{xx}$ is small, say of order $\varepsilon \ll 1$. More precisely, ε is the largest of the ratios of the Navier–Stokes terms (e.g., $D_T k^2$) or $(c_V/T)^{1/2}$ (dT/dz) (i.e., the rate of convective heating by flow perturbations) with respect to ck. Furthermore $\bar{\mathsf{H}}_{yx}$ is of order ε. This observation allows us to identify the two groups of variables, $\bar{x}_k$ and $\bar{y}_k$, as slow and fast variables, respectively.[26] There remains, however, a strong dynamical coupling between the slow and the fast variables because H_{xy} is of order unity. This means physically that the slow variables (e.g., $\bar{T}_\mathbf{k}$) are forced to participate in the fast motion (of $\bar{p}_\mathbf{k}$, e.g.), although the slow variables hardly influence the fast ones.

6. TIME-SCALE PERTURBATION THEORY

It is possible[26] to diminish the coupling between the slow and the fast variables by applying a transformation $\bar{\mathsf{T}}$

$$\begin{pmatrix} \bar{\mathbf{x}}'_\mathbf{k} \\ \bar{\mathbf{y}}'_\mathbf{k} \end{pmatrix} = \bar{\mathsf{T}} \begin{pmatrix} \bar{\mathbf{x}}_\mathbf{k} \\ \bar{\mathbf{y}}_\mathbf{k} \end{pmatrix} \tag{6.1}$$

which can be chosen such that the new dynamical matrix $\bar{\mathsf{H}}' = \bar{\mathsf{T}} \cdot \bar{\mathsf{H}} \cdot \bar{\mathsf{T}}^{-1}$ has the form

$$\bar{\mathsf{H}}' = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \bar{\mathsf{F}}' \end{pmatrix} + \varepsilon \begin{pmatrix} \bar{\mathsf{S}}'_{xx} & \bar{\mathsf{S}}'_{xy} \\ \bar{\mathsf{S}}'_{yx} & \bar{\mathsf{S}}'_{yy} \end{pmatrix} \tag{6.2}$$

where $\bar{\mathsf{F}}'$, $\bar{\mathsf{S}}'_{xx}$,..., $\bar{\mathsf{S}}'_{yy}$ are all of the order ck. The coupling between fast and slow variables is now not anymore of order 1, but of order $\varepsilon \ll 1$. The transformation reads

$$\mathsf{T} = \begin{pmatrix} \mathbf{1} & \bar{\mathsf{c}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix}, \qquad \bar{\mathsf{T}}^{-1} = \begin{pmatrix} \mathbf{1} & -\bar{\mathsf{c}} \\ \mathbf{0} & \mathbf{1} \end{pmatrix} \tag{6.3}$$

with

$$\bar{\mathsf{c}} = \begin{pmatrix} -\left(\dfrac{\gamma - 1}{\gamma}\right)^{1/2} & 0 \\ 0 & 0 \end{pmatrix} \tag{6.4}$$

or, in terms of the unscaled variables

$$T'_{\mathbf{k}} = T_{\mathbf{k}} - \frac{\alpha T}{\rho c_p} p_{\mathbf{k}}, \qquad v'_{\mathbf{k}} = v_{\mathbf{k}}$$
$$p'_{\mathbf{k}} = p_{\mathbf{k}}, \qquad \phi'_{\mathbf{k}} = \phi_{\mathbf{k}} \tag{6.5}$$

The new slow variable $T'_{\mathbf{k}}$ is proportional to the entropy $S_{\mathbf{k}}$ per unit mass: $T'_{\mathbf{k}} = (T/c_p)\, S_k$, which is well known to be true when one linearizes around thermal equilibrium.

In order to diagonalize partially $\bar{\mathsf{H}}$ we will now apply a time-scale perturbation theory, following Geigenmüller *et al.*[26] This will allow us to derive decoupled systems of equations that determine the slow and fast eigenvalues separately. These eigenvalues describe the motion of the slow variables on the slow time scale and that of the fast variables on the fast time scale, and give systematic corrections to both on slower and slower time scales, corresponding to higher and higher orders in ε. We will now outline the procedure.

First we observe that the matrix

$$\bar{\mathsf{F}}' = \begin{pmatrix} 0 & -ck \\ ck & 0 \end{pmatrix} \tag{6.6}$$

has the eigenvalues $+ick$ and $-ick$. Thus, $\bar{\mathsf{H}}'(\mathbf{k})$ has two large eigenvalues of the order ck and two small ones of the order $\varepsilon \cdot ck$. Denoting the small eigenvalues of $\bar{\mathsf{H}}'(\mathbf{k})$ by $s_{\mathbf{k}2}, s_{k3}$ and the large ones by $s_{\mathbf{k}4}, s_{\mathbf{k}5}$ we write the time-evolution operator formally as

$$e^{-\bar{\mathsf{H}}'(\mathbf{k})t} = \sum_{j=2,3} e^{-s_{\mathbf{k}j}t} \bar{\mathsf{A}}^R_{kj} \bar{\mathsf{A}}^{L*}_{kj} + \sum_{j=4,5} e^{-s_{\mathbf{k}j}t} \bar{\mathsf{A}}^R_{kj} \bar{\mathsf{A}}^{L*}_{kj} \tag{6.7}$$

where $\bar{\mathsf{A}}^R_{kj}$ and $\bar{\mathsf{A}}^L_{kj}$ are the normalized right and left eigenvectors of $\bar{\mathsf{H}}'(\mathbf{k})$, respectively. The second term in (6.7) describes rapid oscillations which average to zero on the slow time scale $\tau_{\mathrm{sl}} = (1/\varepsilon)\, \tau_{\mathrm{fa}}$.

Define by

$$\bar{\mathsf{P}} = \sum_{j=1,2} \bar{\mathbf{A}}^R_{kj} \bar{\mathbf{A}}^{L*}_{kj} \tag{6.8}$$

the projector on the "slow" subspace, i.e., the space spanned by the eigenvectors corresponding to the small eigenvalues. On the slow time scale τ_{sl} only solutions $\bar{\mathbf{A}}'_{\mathbf{k}} = (\bar{\mathbf{x}}'_{\mathbf{k}}, \bar{\mathbf{y}}'_{\mathbf{k}})$ satisfying

$$\bar{\mathbf{A}}'_{\mathbf{k}} = \bar{\mathbf{P}}_{\mathbf{k}} \cdot \bar{\mathbf{A}}'_{\mathbf{k}} \tag{6.9}$$

contribute to the dynamics. Using that $\bar{\mathsf{P}}^2 = \bar{\mathsf{P}}$, Eq. (6.9) allows us to express the fast variables formally in terms of the slow ones as

$$\bar{\mathbf{y}}'_{\mathbf{k}} = \bar{\mathsf{R}}'_{\mathrm{sl}} \cdot \bar{\mathbf{x}}'_{\mathbf{k}} \tag{6.10}$$

where $\bar{\mathsf{R}}'_{\mathrm{sl}} = \bar{\mathsf{P}}_{yx} \cdot \bar{\mathsf{P}}^{-1}_{xx}$ is a so-called reconstruction operator. Inserting (6.10) into the transformed dynamical equations (5.20) one can eliminate $\bar{\mathbf{y}}'_{\mathbf{k}}$ and derive a reduced dynamical equation

$$\frac{\partial}{\partial t} \bar{\mathbf{x}}'_{\mathbf{k}} = -\bar{\mathsf{H}}'_{\mathrm{sl}}(\mathbf{k}) \cdot \bar{\mathbf{x}}'_{\mathbf{k}} \tag{6.11}$$

which describes the motion of the slow variables on the slow time scale. For the reduced dynamical matrix one obtains then

$$\bar{\mathsf{H}}'_{\mathrm{sl}} = \varepsilon(\bar{\mathsf{S}}'_{xx} + \bar{\mathsf{S}}'_{xy} \cdot \bar{\mathsf{R}}'_{\mathrm{sl}}) \tag{6.12}$$

Inserting (6.10)–(6.12) into (5.20) one obtains also an equation for the reconstruction operator:

$$\varepsilon \bar{\mathsf{R}}'_{\mathrm{sl}} \cdot (\bar{\mathsf{S}}'_{xx} + \bar{\mathsf{S}}'_{xy} \cdot \bar{\mathsf{R}}'_{\mathrm{sl}}) = \bar{\mathsf{F}}' \cdot \bar{\mathsf{R}}'_{\mathrm{sl}} + \varepsilon \bar{\mathsf{S}}'_{yx} + \varepsilon \bar{\mathsf{S}}'_{yy} \cdot \bar{\mathsf{R}}'_{\mathrm{sl}} \tag{6.13}$$

This equation can be solved iteratively in powers of ε:

$$\bar{\mathsf{R}}'_{\mathrm{sl}} = -\varepsilon(\bar{\mathsf{F}}')^{-1} \cdot \bar{\mathsf{S}}'_{yx} + \mathcal{O}(\varepsilon^2) \tag{6.14}$$

In a similar way one can ask for the solutions $\bar{\mathbf{A}}'_{\mathbf{k}}$ of the dynamical equation which lie in the "fast" subspace, i.e., the space spanned by the eigenvectors corresponding to the large eigenvalues:

$$\bar{\mathbf{A}}'_{\mathbf{k}} = \bar{\mathsf{Q}} \cdot \bar{\mathbf{A}}'_{\mathbf{k}} \tag{6.15}$$

where $\bar{\mathsf{Q}} = \mathbf{1} - \bar{\mathsf{P}}$. Equation (6.15) allows us to eliminate the slow variables

$$\bar{\mathbf{x}}'_{\mathbf{k}} = \bar{\mathsf{R}}'_{\mathrm{fa}} \cdot \bar{\mathbf{y}}'_{\mathbf{k}} \tag{6.16}$$

where the reconstruction operator is formally defined by $\bar{\mathsf{R}}'_{\mathrm{fa}} = \bar{\mathsf{Q}}_{xy} \cdot \bar{\mathsf{Q}}^{-1}_{yy}$. Inserting (6.15) into the dynamical equations, one derives the reduced equation

$$\frac{\partial}{\partial t} \bar{\mathbf{y}}'_{\mathbf{k}} = -\bar{\mathsf{H}}'_{\mathrm{fa}}(\mathbf{k}) \cdot \bar{\mathbf{y}}'_{\mathbf{k}} \tag{6.17}$$

which describes the motion of the fast variables. The reduced dynamical matrix is

$$\bar{\mathsf{H}}'_{\mathrm{fa}} = \bar{\mathsf{F}}' + \varepsilon(\bar{\mathsf{S}}'_{yy} + \bar{\mathsf{S}}'_{yx} \cdot \bar{\mathsf{R}}'_{\mathrm{fa}}) \tag{6.18}$$

and $\bar{\mathsf{R}}'_{\mathrm{fa}}$ follows from

$$\bar{\mathsf{R}}'_{\mathrm{fa}} \cdot (\bar{\mathsf{F}}' + \varepsilon \bar{\mathsf{S}}'_{yy} + \varepsilon \bar{\mathsf{S}}'_{yx} \cdot \bar{\mathsf{R}}'_{\mathrm{fa}}) = \varepsilon \bar{\mathsf{S}}'_{xx} \cdot \bar{\mathsf{R}}'_{\mathrm{fa}} + \varepsilon \bar{\mathsf{S}}'_{xy} \tag{6.19}$$

Solving (6.19) iteratively yields

$$\bar{\mathsf{R}}'_{\mathrm{fa}} = \varepsilon \bar{\mathsf{S}}'_{xy} \cdot (\bar{\mathsf{F}}')^{-1} + \mathcal{O}(\varepsilon^2) \tag{6.20}$$

Equations (6.11) and (6.17) are axact when the reconstruction operators are computed to all orders in ε. Since ε is very small for wavelengths in the regime (5.24) it is sufficient to determine the reduced matrices up to first order in ε. In this approximation the reconstruction operators do not contribute to $\bar{\mathsf{H}}'_{\mathrm{sl}}$ and $\bar{\mathsf{H}}'_{\mathrm{fa}}$.

Transforming Eqs. (6.11) and (6.17) back to the unscaled variables $\mathbf{x}'_k$ and $\mathbf{y}'_k$, defined by (6.5), we find for the dynamical matrices to order ε

$$\mathsf{H}'_{\mathrm{sl}}(\mathbf{k}) = \begin{pmatrix} D_T k^2 & \dfrac{dT}{dz} \\ -\alpha g \dfrac{k_\parallel^2}{k^2} & \nu k^2 \end{pmatrix} \tag{6.21}$$

and[9]

$$\mathsf{H}'_{\mathrm{fa}}(\mathbf{k}) = \begin{pmatrix} (\gamma - 1)\, D_T k^2 & -\dfrac{\gamma}{\chi_T} k^2 \\ \dfrac{1}{\rho} & \Gamma_l k^2 \end{pmatrix} \tag{6.22}$$

The eigenvalues $s_{\mathbf{k}2}, \ldots, s_{\mathbf{k}5}$ can now easily be obtained. Those of (6.21), i.e., the small eigenvalues, are

$$s_{\mathbf{k}2,3} = \frac{\nu + D_T}{2} k^2 \pm \frac{\nu - D_T}{2} k^2 \left[1 - \frac{4\alpha g}{(\nu - D_T)^2} \frac{dT}{dz} \frac{k_\parallel^2}{k^6} \right]^{1/2} \tag{6.23}$$

In equilibrium ($dT/dz = 0$), (6.23) reduces to $s_{\mathbf{k},2} = \nu k^2$ and $s_{\mathbf{k},3} = D_T k^2$. These are the familiar eigenvalues of the viscous and the heat mode, respectively. In nonequilibrium, according to (6.23), there is a strong coupling between these modes (especially for small k), and we will therefore call them the viscoheat modes.

The large eigenvalues, i.e., those of (6.22), are the sound modes. Treating the terms of order $\varepsilon \cdot ck$, i.e., $(\gamma - 1)\, D_T k^2$ and $\Gamma_l k^2$, in perturbation theory one finds

$$s_{\mathbf{k}4,5} = \pm ick + \tfrac{1}{2}\Gamma_s k^2 \tag{6.24}$$

[9] For the wavelengths considered here [cf. Eq. (5.24)] we can neglect the gravity terms in $\bar{\mathsf{H}}_{yy}$ [cf. Eq. (5.22)] and, hence, also in $\mathsf{H}'_{\mathrm{fa}}$.

where

$$\Gamma_s = \Gamma_l + (\gamma - 1)\, D_T \tag{6.25}$$

is the sound absorption coefficient. (6.24) has the same form as in equilibrium.

To complete our "local" eigenvalue analysis we have finally to verify the basic condition (5.11) which restricts the mean-free path of a given mode to lengths such that the spatial variation of the average quantities can be neglected. For those modes for which (5.11) is violated we have to supplement (5.7) by a further restriction on the wave vector. First, the mean-free path of the viscous mode is zero, since its eigenvalue $s_{\mathbf{k}1}$ [cf. Eq. (5.18)] is real. Next, the viscoheat modes can have a nonvanishing mean-free path for $dT/dz > 0$ (heating from above). However, the real and imaginary parts of the eigenvalues $s_{\mathbf{k}2,3}$ are then both of the order $\varepsilon \cdot ck$ and, hence, $l_{2,3}(\mathbf{k}) \approx 1/k \ll L_\nabla$. Finally, for the sound modes one has $l_{4,5}(\mathbf{k}) = 2c/\Gamma_s k^2$. To satisfy (5.11) we have thus to require that

$$k \gg \left(\frac{2c}{\Gamma_s L_\nabla} \cdot \frac{k_z}{k}\right)^{1/2} \qquad \text{(sound)}. \tag{6.26}$$

This restriction is actually rather severe.[10] Therefore the method used in the last two sections to obtain the eigenvalues applies to the viscous and the viscoheat modes, but to the sound modes only as long as (6.26) holds. This excludes the physically interesting case of sound modes that pass through a number of layers before they are damped. In addition, boundary effects, which have been neglected so far, may become important for all modes. We will take these problems into account in the next section.

7. NONEQUILIBRIUM MODES

We are now in a position to reduce the general spectral problem (5.1) of the hydrodynamic operator $\mathfrak{H}(\mathbf{r})$ in real space (so that boundary conditions can be applied) to three separate eigenvalue problems for the viscous, the viscoheat, and the sound modes, respectively. In this section we will give the eigenvalue equations leaving their explicit solutions with the appropriate boundary conditions to papers II and III.

From translational invariance in the x–y plane follows that the eigenvectors $\mathbf{a}^R_K(\mathbf{r})$ and $\mathbf{a}^L_K(\mathbf{r})$ of $\mathfrak{H}(\mathbf{r})$ are plane waves in the x and y directions characterized by some horizontal wave vector $\mathbf{k}_\parallel = (k_x, k_y)$. As in the

[10] For example, for water under normal conditions and a temperature gradient $dT/dz = 50$ K cm^{-1} one has $L_\nabla \simeq 6$ cm and, thus, $2c/\Gamma_s L_\nabla \approx 2000$ cm^{-1}.

previous sections we will restrict ourselves to those modes which vary in the z direction on length scales $l_0 \ll L_\nabla$. Then a condition like (5.7) holds still in the presence of boundaries if the wave number k_z is understood in a more qualitative sense as a measure for the periodicity of the mode in the z direction. The eigenvalue equations are then simply found by translating the results of the last section into real space, and replacing ik_z by d/dz. This is not true for the sound modes, however, unless the size of the system d is much smaller than L_∇. If d does not satisfy this condition, the results derived in this way are only valid for wave vectors large enough to obey the condition (6.26), and one must in general take into account explicitly the variation of the average quantities with position.

7.1. Viscous Modes

We obtain the right-eigenvalue equation for the viscous modes for a given horizontal wave vector $\mathbf{k}_\parallel$ with (5.17):

$$\nu\mathscr{D}\xi^R_{\nu,k_\parallel n}(z) = s_{\nu,k_\parallel n}\xi^R_{\nu,k_\parallel n}(z) \tag{7.1}$$

In (7.1) we have introduced the operator

$$\mathscr{D} = k_\parallel^2 - \frac{d^2}{dz^2} \tag{7.2}$$

The index ν serves to indicate that we are dealing with the class of viscous modes; the index n, which is discrete because of the boundary conditions, labels the different viscous modes. The eigenvalues $s_{\nu,k_\parallel n}$ and eigenfunctions $\xi^R_{\nu,k_\parallel n(z)}$ depend only on $k_\parallel = |\mathbf{k}_\parallel|$, owing to rotational invariance in the x–y plane. The right eigenvector $\mathbf{a}^R_{\nu,\mathbf{k}_\parallel n}$ can be expressed solely in terms of the vorticity $\xi_{\nu,k_\parallel n}$ since all other variables are not involved. Similarly, the left eigenvector $\mathbf{a}^L_{\nu,\mathbf{k}_\parallel n}$ is determined by the left eigenfunction $\xi^L_{\nu,k_\parallel n}$ which follows also from (7.1) since this equation is self-adjoint. We will give the results for the eigenvectors in terms of the old variables $\mathbf{a} = (p, T, \mathbf{u})$. The flow velocity $\mathbf{u} = (u_x, u_y, u_z)$ in Cartesian components is obtained from (5.15). Then

$$\mathbf{a}^R_{\nu,\mathbf{k}_\parallel n}(\mathbf{r}) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{k_\parallel^2}(\nabla \times \mathbf{e}_z)\, \xi^R_{\nu,k_\parallel n}(z) \end{pmatrix} e^{i\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel} \tag{7.3a}$$

$$\mathbf{a}^L_{\nu,\mathbf{k}_\parallel n}(\mathbf{r}) = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{k_\parallel^2}(\nabla \times \mathbf{e}_z)\, \xi^L_{\nu,k_\parallel n}(z) \end{pmatrix} e^{i\mathbf{k}_\parallel \cdot \mathbf{r}_\parallel} \tag{7.3b}$$

respectively, where $\mathbf{r}_\| = (x, y)$. Note that the differential operators act also on the exponential factors.

7.2. Viscoheat Modes

From the four variables (6.5) only the slow ones, i.e., $T'_\mathbf{k}$ and $v'_\mathbf{k}$, are involved in the viscoheat modes. Translating Eq. (6.21) for $\mathsf{H}'_{sl}(\mathbf{k})$ into real space and using that the pressure component is zero, we can write the coupled right-eigenvalue equations in terms of the untransformed temperature and the transversal velocity as

$$\begin{pmatrix} D_T \mathscr{D} & \dfrac{dT}{dz} \\ -\alpha g k_\|^2 & \nu \mathscr{D}^2 \end{pmatrix} \begin{pmatrix} T^R_{\lambda,k_\| n}(z) \\ v^R_{\lambda,k_\| n}(z) \end{pmatrix} = s_{\lambda,k_\| n} \begin{pmatrix} T^R_{\lambda,k_\| n}(z) \\ \mathscr{D} v^R_{\lambda,k_\| n}(z) \end{pmatrix} \tag{7.4}$$

where λ denotes the class of viscoheat modes and n is a discrete index labeling the different viscoheat modes. Using (6.5) and (5.15) the right and left eigenvectors are

$$\mathbf{a}^R_{\lambda,\mathbf{k}_\| n}(\mathbf{r}) = \begin{pmatrix} 0 \\ T^R_{\lambda,k_\| n}(z) \\ \dfrac{1}{k_\|^2} \nabla \times (\nabla \times \mathbf{e}_z)\, v^R_{\lambda,k_\| n}(z) \end{pmatrix} e^{i\mathbf{k}_\| \cdot \mathbf{r}_\|} \tag{7.5a}$$

$$\mathbf{a}^L_{\lambda,\mathbf{k}_\| n}(\mathbf{r}) = \begin{pmatrix} -\dfrac{\alpha T}{\rho c_p} T^L_{\lambda,k_\| n}(z) \\ T^L_{\lambda,k_\| n}(z) \\ \dfrac{1}{k_\|^2} \nabla \times (\nabla \times \mathbf{e}_z)\, v^L_{\lambda,k_\| n}(z) \end{pmatrix} e^{i\mathbf{k}_\| \cdot \mathbf{r}_\|} \tag{7.5b}$$

The left eigenfunctions $(T^L_{\lambda,k_\| n}, v^L_{\lambda,k_\| n})$ follow from the adjoint eigenvalue problem of (7.4). In deriving (7.5b) we have used that the adjoint vectors $\mathbf{a}^\dagger$ transform with the matrix $(\mathsf{T}^{-1})^\dagger$ when T is the transformation given by (6.5). Thus the transformed variables are $T^{\dagger\prime} = T^\dagger$, $v^{\dagger\prime} = v^\dagger$, $p^{\dagger\prime} = p^\dagger + (\alpha T/\rho c_p)\, T^\dagger$, $\phi^{\dagger\prime} = \phi^\dagger$. Setting $p^\dagger = 0$, $\phi^\dagger = 0$, one obtains $\mathbf{a}^L_{\lambda,\mathbf{k}_\| n}$.

As in the previous section, the values of the steady state quantities in formulas (7.1), (7.3)–(7.5) must be taken in a reference point R_z on the z axis in the center of the layer of height l_0 under consideration and their spatial variation can be neglected therein. In the case $d \ll L_\nabla$ it is natural to choose the center $R_z = 0$ as the reference point.

7.3. Sound Modes

As was noticed before, we may in general not simply translate the matrix $\mathsf{H}'_{\mathrm{fa}}$ into real space in order to find the equations for the sound modes, unless $d \ll L_\nabla$. Especially waves propagating almost in the z direction ($k \approx k_z$) may cross several fluid layers of height l_0 before they are damped. These modes will "feel" also the gradients of the average quantities. However, since the classification of the hydrodynamic variables into fast and slow ones holds for each layer in the fluid, it is also globally valid for the whole fluid.[11]

But we have to give up the assumption that the average quantities can effectively be treated as constants. To find the equations for the sound modes, we therefore start from the full hydrodynamic operator given by (4.5)–(4.8), substitute the transformed variables (6.5) into the equations and set the slow variables equal to zero. The resulting right-eigenvalue equations for the sound modes read then

$$\begin{pmatrix} -\dfrac{\gamma-1}{\alpha T}\nabla^2\lambda\dfrac{\alpha T}{\rho c_p} & \dfrac{\gamma}{\chi_T}\nabla^2 \\ \nabla\cdot\dfrac{1}{\rho}\nabla & -\nabla\dfrac{1}{\rho}\nabla\colon\left[2\eta\nabla\nabla+\left(\zeta-\dfrac{2}{3}\eta\right)\nabla^2\mathbf{1}\right] \end{pmatrix}\begin{pmatrix} p^R_{\pm,k_\| n}(z) \\ \phi^R_{\pm,k_\| n}(z)\end{pmatrix} e^{i\mathbf{k}_\|\cdot\mathbf{r}_\|}$$

$$= s_{\pm,k_\| n}\begin{pmatrix} p^R_{\pm,k_\| n}(z) \\ \nabla^2\phi^R_{\pm,k_\| n}(z)\end{pmatrix} e^{i\mathbf{k}_\|\cdot\mathbf{r}_\|} \tag{7.6}$$

where the values of the steady state quantities have to be taken in the point R_z. The indices $\pm$ denote the two classes of sound modes, in particular the index $+$ denotes the modes, the eigenvalues of which have a positive imaginary part, while $-$ denotes those with negative imaginary part [cf. (6.24)]. In deriving (7.6) we have neglected the extremely weak dependence of the heat conductivity on the pressure (see Ref. 32) and the gravity contribution (see footnote 9). The right and left eigenvectors are

$$\mathbf{a}^R_{\pm,\mathbf{k}_\| n}(\mathbf{r}) = \begin{pmatrix} p^R_{\pm,k_\| n}(z) \\ \dfrac{\alpha T}{\rho c_p}\, p^R_{\pm,k_\| n}(z) \\ \nabla\phi^R_{\pm,k_\| n}(z) \end{pmatrix} e^{i\mathbf{k}_\|\cdot\mathbf{r}_\|} \tag{7.7a}$$

[11] In practice, the cur-off for the mean-free path, given by the finite d, excludes the pathological case that the change of the average quantities over the mean-free path is so large that the global classification into fast and slow variables breaks down. This pathological case would require that $d \gg L_\nabla$. However, under this condition the system is too far from equilibrium to stay in a laminar stationary state with $\mathbf{u}=0$.

$$\mathbf{a}^L_{\pm,\mathbf{k}_\parallel n}(\mathbf{r}) = \begin{pmatrix} p^L_{\pm,k_\parallel n}(z) \\ 0 \\ \nabla\phi^L_{\pm,k_\parallel n}(z) \end{pmatrix} e^{i\mathbf{k}_\parallel\cdot\mathbf{r}_\parallel} \tag{7.7b}$$

After all the eigenvalue equations (7.1), (7.4), and (7.6) have been solved with the appropriate boundary conditions, the eigenvectors (7.3), (7.5), and (7.7) must finally be normalized according to (5.2). The normalization conditions are

$$\int_{-d/2}^{d/2} \xi^{L*}_{v,k_\parallel n}\xi^R_{v,k_\parallel m}\,dz = \frac{k_\parallel^2}{(2\pi)^2}\delta_{nm} \tag{7.8a}$$

for the viscous modes,

$$\int_{-d/2}^{d/2}\left[v^{L*}_{\lambda,k_\parallel n}v^R_{\lambda,k_\parallel m} + \frac{1}{k_\parallel^2}\frac{dv^L_{\lambda,k_\parallel n}}{dz}\frac{dv^R_{\lambda,k_\parallel m}}{dz} + T^{L*}_{\lambda,k_\parallel n}T^R_{\lambda,k_\parallel m}\right]dz$$
$$= \frac{1}{(2\pi)^2}\delta_{nm} \tag{7.8b}$$

for the viscoheat modes, and

$$\int_{-d/2}^{d/2}\left[k_\parallel^2\phi^{L*}_{\sigma,k_\parallel n}\phi^R_{\sigma',k_\parallel m} + \frac{d\phi^{L*}_{\sigma,k_\parallel n}}{dz}\frac{d\phi^R_{\sigma',k_\parallel m}}{dz} + p^{L*}_{\sigma,k_\parallel n}p^R_{\sigma',k_\parallel m}\right]dz$$
$$= \frac{1}{(2\pi)^2}\delta_{nm}\,\delta_{\sigma\sigma'} \qquad (\sigma,\sigma' = +,-) \tag{7.8c}$$

for the sound modes, respectively.

8. CALCULATION OF THE CORRELATION MATRIX

The results of the previous sections allow us to derive more detailed expressions for the 11 scalar functions $g_{pp},\ldots,h_4$ which have been introduced in (2.9) from symmetry arguments alone. For this purpose we apply our nonequilibrium hydrodynamic modes to solve the equations (3.7) and (4.3) for the correlation matrix, that have been derived from fluctuating hydrodynamics. It is clear from the assumptions we have made above in deriving the eigenvalue equations that the results will then be valid quite generally for all distances $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$ and all time intervals $t=t_1-t_2$ of hydrodynamic order, the only restriction being that $|z_1-z_2|\lesssim l_0 \ll L_\nabla$.

Thus we use (5.1)–(5.3) to solve Eq. (3.7) formally for $t_1 \geqslant t_2$:

$$\mathsf{M}(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2)=\sum_K e^{-s_K(t_1-t_2)}\mathbf{a}^R_K(\mathbf{r}_1)\,\mathbf{b}_K(\mathbf{r}_2) \qquad (t_1\geqslant t_2) \tag{8.1}$$

where

$$\mathbf{b}_K(\mathbf{r}_2) = \int \mathbf{a}_K^{L*}(\mathbf{r}_1) \cdot \mathsf{M}(\mathbf{r}_1, t_2; \mathbf{r}_2, t_2)\, d\mathbf{r}_1 \tag{8.2}$$

Inserting (4.1)–(4.3) into (8.2) and using again (5.1)–(5.3) yields

$$\mathbf{b}_K(\mathbf{r}_2) = \mathsf{A}^{(0)}(\mathbf{r}_2) \cdot \mathbf{a}_K^{L*}(\mathbf{r}_2) - \sum_{K'} \frac{B_{KK'}}{s_K + s_{K'}} \mathbf{a}_{K'}^{R}(\mathbf{r}_2) \tag{8.3}$$

with the mode-coupling coefficients $B_{KK'}$ given by

$$B_{KK'} = \iint \mathsf{B}(\mathbf{r}_1, \mathbf{r}_2) \colon \mathbf{a}_K^{L*}(\mathbf{r}_1)\, \mathbf{a}_{K'}^{L*}(\mathbf{r}_2)\, d\mathbf{r}_1\, d\mathbf{r}_2 \tag{8.4}$$

The contribution of the local equilibrium part A is represented in (8.1) as a sum over all the single eigenmodes, while the long-range part D is expressed as a double sum over all pairs of eigenmodes, K and K', which are coupled via the $B_{KK'}$.

The mode-coupling coefficients $B_{KK'}$ consist of three contributions, according to (4.16). Their orders of magnitude can easily be estimated using the scaled variables (5.19) and that $\bar{a}_k^{L*} \sim e^{-ik \cdot r}$. Then one finds that $\Lambda_i^{(1)}$ is of order $\varepsilon \ll 1$ relative to C, while $\Lambda^{(2)}$ is even of order $\varepsilon / L_\nabla k_z \ll \varepsilon$. Keeping only the leading order in ε in the amplitude of the correlation matrix we can neglect the terms $\Lambda_i^{(1)}$ and $\Lambda^{(2)}$ and set $\mathsf{B}(\mathbf{r}_1, \mathbf{r}_2) \approx \mathsf{C}(\mathbf{r}_1, \mathbf{r}_2)$.[12] Using (4.12) the mode-coupling coefficients are then

$$B_{KK'} = \int \mathsf{C}^{(0)}(\mathbf{r}) \colon \mathbf{a}_K^{L*}(\mathbf{r})\, \mathbf{a}_{K'}^{L*}(\mathbf{r})\, d\mathbf{r} \tag{8.5}$$

Inserting (4.13) and the left eigenvectors (7.3b) into (8.5), we find that $B_{KK'} = 0$ if one of the modes is a viscous mode. Furthermore we see from (8.3) that the leading contributions to $\mathbf{b}_k$ come from those combinations of modes where $s_K + s_{K'}$ is small. Recalling (6.23), (6.24), one realizes that there are only two such combinations: (i) two viscoheat modes, since both their eigenvalues are of order $\varepsilon \cdot ck$, and (ii) a $+$ and a $-$ sound mode, or vice versa, since the large terms of order ck cancel. Keeping again only the leading order in ε, we can ignore in (8.3) the remaining couplings between the modes $++$, $--$, $\lambda+$, and $\lambda-$.

On the basis of (8.1), (8.3), and (8.5), the correlation matrix can now be formally expressed in terms of the eigenvalues and the normalized eigenfunctions of the viscous, the viscoheat, and the sound modes. Using the

[12] This proves the statement made at the end of Section 4.

explicit expressions (4.13), (7.5b), and (7.7b), the relevant mode-coupling coefficients $B_{KK'}$ arising from the + and − sound modes are

$$\Pi_{+n,-m} = \int_{-d/2}^{+d/2} \frac{\gamma}{\rho\chi_T} \frac{dT}{dz} \left[p^{L*}_{+,k_\| n} \frac{d\phi^{L*}_{-,k_\| m}}{dz} + \frac{d\phi^{L*}_{+,k_\| n}}{dz} p^{L*}_{-,k_\| m} \right] dz \quad (8.6a)$$

while those from the viscoheat modes are

$$\Lambda_{nm} = \int_{-d/2}^{+d/2} \frac{T}{\rho} \frac{dT}{dz} \left[T^{L*}_{\lambda,k_\| n} v^{L*}_{\lambda,k_\| m} + v^{L*}_{\lambda,k_\| n} T^{L*}_{\lambda,k_\| m} \right] dz \quad (8.6b)$$

Inserting (3.10), (7.5), and (7.7) into (8.3) we introduce

$$\pi_{\pm,k_\| n}(z_2) = \frac{\gamma T}{\chi_T} p^{L*}_{\pm,k_\| n}(z_2) - (2\pi)^2 \sum_m \frac{\Pi_{\pm n,\mp m}}{s_{\pm,k_\| n} + s_{\mp,k_\| m}} p^R_{\mp,k_\| m}(z_2)$$

$$\psi_{\pm,k_\| n}(z_2) = \frac{T}{\rho} \phi^{L*}_{\pm,k_\| n}(z_2) - (2\pi)^2 \sum_m \frac{6_{\pm n,\mp m}}{s_{\pm,k_\| n} + s_{\mp,k_\| m}} \phi^R_{\mp,k_\| m}(z_2) \quad (8.7a)$$

$$\theta_{\lambda,k_\| n}(z_2) = \frac{T^2}{\rho c_p} T^{L*}_{\lambda,k_\| n}(z_2) - (2\pi)^2 \sum_m \frac{\Lambda_{nm}}{s_{\lambda,kn} + s_{\lambda,k_\| m}} T^R_{\lambda,k_\| m}(z_2)$$

$$w_{\lambda,k_\| n}(z_2) = \frac{T}{\rho} v^{L*}_{\lambda,k_\| n}(z_2) - (2\pi)^2 \sum_m \frac{\Lambda_{nm}}{s_{\lambda,k_\| n} + s_{\lambda,k_\| m}} v^R_{\lambda,k_\| m}(z_2) \quad (8.7b)$$

Using these functions one can compute M, or equivalently, the scalar functions $g_{pp}, \ldots, h_4$ according to (8.1). It is convenient to express M in terms of correlation functions which are generated either by the sound modes only or by the viscoheat modes only. Thus we set

$$P(r_\|, z_1, z_2; t) = k_B 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) \sum_{\sigma=+,-} e^{-s_{\sigma,k_\| n} t} p^R_{\sigma,k_\| n}(z_1) \pi_{\sigma,k_\| n}(z_2)$$

$$P'(r_\|, z_1, z_2; t) = k_B 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) \sum_{\sigma=+,-} e^{-s_{\sigma,k_\| n} t} p^R_{\sigma,k_\| n}(z_1) \psi_{\sigma,k_\| n}(z_2)$$

$$\Phi(r_\|, z_1, z_2; t) = k_B 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) \sum_{\sigma=+,-} e^{-s_{\sigma,k_\| n} t} \phi^R_{\sigma,k_\| n}(z_1) \psi_{\sigma,k_\| n}(z_2)$$

for the soundmodes and (8.8a)

$$S(r_\|, z_1, z_2; t) = k_B \frac{c_p^2}{T^2} 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) e^{-s_{\lambda,k_\| n} t} T^R_{\lambda,k_\| n}(z_1) \theta_{\lambda,k_\| n}(z_2)$$

$$S'(r_\|, z_1, z_2; t) = k_B \frac{c_p}{T} 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) \frac{e^{-s_{\lambda,k_\| n} t}}{k_\|^2} T^R_{\lambda,k_\| n}(z_1) w_{\lambda,k_\| n}(z_2)$$

$$V(r_\|, z_1, z_2; t) = k_B 2\pi \sum_n \int_0^\infty dk_\| k_\| J_0(k_\| r_\|) \frac{e^{-s_{\lambda,k_\| n} t}}{k_\|^4} v^R_{\lambda,k_\| n}(z_1) w_{\lambda,k_\| n}(z_2)$$

(8.8b)

for the viscoheat modes. In addition, there is a contribution to (8.1) from the viscous modes. Therefore we also introduce

$$\Xi(r_{\|}, z_1, z_2; t) = k_B \frac{T}{\rho} 2\pi \sum_n \int_0^\infty dk_{\|} k_{\|} J_0(k_{\|} r_{\|}) \frac{e^{-s_{v,k_{\|}n}t}}{k_{\|}^4} \xi^R_{v,k_{\|}n}(z_1)\, \xi^{L*}_{v,k_{\|}n}(z_2) \tag{8.8c}$$

(8.8c) does not contain mode-coupling contributions. In (8.8) $J_0(x)$ is the Bessel function of order zero. Furthermore P and S are just the pressure–pressure and the entropy–entropy correlation functions, respectively, while Ξ is related to the vorticity–vorticity correlation function. We note that the functions P, P', and Φ average to zero on the slow time scale τ_{sl}. Using (8.8) we find in a straightforward manner:

$$g_{pp} = P, \qquad g_{pT} = \frac{\alpha T}{\rho c_p} P$$

$$g_{TT} = \frac{T^2}{c_p^2} S + \left(\frac{\alpha T}{\rho c_p}\right)^2 P \tag{8.9a}$$

$$f_{p1} = \frac{\partial}{\partial z_2} P', \qquad f_{p2} = -\frac{\partial}{\partial r_{\|}} P'$$

$$f_{T1} = \frac{T}{c_p}\left[\frac{\alpha}{\rho} f_{p1} - \left(\frac{\partial}{\partial r_{\|}} + \frac{1}{r_{\|}}\right)\frac{\partial}{\partial r_{\|}} S'\right]$$

$$f_{T2} = \frac{T}{c_p}\left[\frac{\alpha}{\rho} f_{p2} - \frac{\partial}{\partial r_{\|}}\frac{\partial}{\partial z_2} S'\right] \tag{8.9b}$$

$$h_1 = -\frac{\partial^2}{\partial r_{\|}^2}\Xi - \frac{1}{r_{\|}}\frac{\partial}{\partial r_{\|}}\left(\Phi + \frac{\partial^2}{\partial z_1 \partial z_2} V\right)$$

$$h_2 = \frac{\partial^2}{\partial z_1 \partial z_2}\Phi + \left[\left(\frac{\partial}{\partial r_{\|}} + \frac{1}{r_{\|}}\right)\frac{\partial}{\partial r_{\|}}\right]^2 V$$

$$h_3 = -\left(\frac{\partial}{\partial r_{\|}} - \frac{1}{r_{\|}}\right)\frac{\partial}{\partial r_{\|}}\left[\Phi - \Xi + \frac{\partial^2}{\partial z_1 \partial z_2} V\right]$$

$$h_4 = -\frac{\partial}{\partial r_{\|}}\left[\frac{\partial}{\partial z_1}\Phi - \left(\frac{\partial}{\partial r_{\|}} + \frac{1}{r_{\|}}\right)\frac{\partial}{\partial r_{\|}}\frac{\partial}{\partial z_2} V\right] \tag{8.9c}$$

In (8.7)–(8.9) the values of the steady state quantities ρ, T, α, χ_T, c_p, γ are to be taken in the reference point R_z of the layer of height l_0 to which z_1 and z_2 belong. From (8.9) and the symmetry relation (2.10) one can find all the elements of the hydrodynamic correlation matrix $\mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ for times $t_1 \geq t_2$ and $t_1 < t_2$.

9. CONCLUSIONS

a. In this paper we have developed a formal theory for the correlation functions $\langle \delta \mathbf{a}(\mathbf{r}_1, t_1)\, \delta \mathbf{a}(\mathbf{r}_2, t_2) \rangle_{ss}$ in a fluid which is exposed to a stationary heat flux in the z direction, parallel to the gravity field. Our results hold for all distances $|\mathbf{r}_1 - \mathbf{r}_2|$ and time intervals $|t_1 - t_2|$ of hydrodynamic order with the restriction that $|z_1 - z_2|$ does not exceed a length l_0 which is small on the macroscopic length scale L_∇.

b. Applying fluctuating hydrodynamics we have argued that the equal-time correlation matrix contains a long-range part which vanishes in equilibrium. These long-range correlations are caused by the nonlinear convection term and by the spatial inhomogenity of the macroscopic variables in the steady state solution around which one linearizes.

c. In solving the equations for the correlation matrix we have made extensive use of the fact that the hydrodynamic operator $\mathfrak{H}(\mathbf{r})$ contains a small parameter $\varepsilon = \tau_{sl}/\tau_{fa} \ll 1$. This allowed us to diagonalize $\mathfrak{H}(\mathbf{r})$ partially into three separate eigenvalue problems for the viscous, the viscoheat, and the sound modes.

d. We have treated the temporal behavior of the correlationfunctions up to first order in ε, so that evolution processes on the fast as well as on the slow timescale are described. The amplitudes have only been taken into account to leading order in ε. To this order in ε there are only couplings between two nonequilibrium sound modes of opposite sign ($+$ and $-$) or between two viscoheat modes. It are only these mode-couplings that contribute to the correlation matrix $\mathsf{M}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$, in addition to the local equilibrium contributions.

e. In papers II and III, the formal expressions for $g_{pp}, \ldots, h_4$ will be evaluated explicitly by solving the equations for the hydrodynamic modes for a number of cases.

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